Problem Set 9 – Relativity (GOI36A)

1 Einstein-Maxwell Theory

1.) The Maxwell Lagrangian can be written as

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \ , \tag{1.1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field strength. $F_{\mu\nu}$ has no explicit dependence on the metric, and so under a small variation $\delta g_{\mu\nu}$ of the metric, the quantity $\sqrt{-g}\,\mathcal{L}_M$ transforms as

$$\delta(\sqrt{-g}\,\mathcal{L}_M) = -\frac{1}{4}\delta\left(\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}\right)F_{\mu\nu}F_{\rho\sigma} \ . \tag{1.2}$$

We showed last week that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$, and so (using the product rule) we can write the variation above as

$$\delta\left(\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}\right) = \sqrt{-g}\left(-\frac{1}{2}g_{\lambda\tau}g^{\mu\rho}g^{\nu\sigma}\delta g^{\lambda\tau} + g^{\nu\sigma}\delta g^{\mu\rho} + g^{\mu\rho}\delta g^{\nu\sigma}\right) \ . \tag{1.3}$$

Plugging this back into (1.2) and contracting indices appropriately, we find that

$$\begin{aligned}\delta(\sqrt{-g}\,\mathcal{L}_M) &= -\frac{1}{4}\sqrt{-g}\left(-\frac{1}{2}g_{\lambda\tau}F_{\mu\nu}F^{\mu\nu}\delta g^{\lambda\tau} + F_{\mu\nu}F_{\rho}{}^{\nu}\delta g^{\mu\rho} + F_{\mu\nu}F^{\mu}{}_{\sigma}\,\delta g^{\nu\sigma}\right)\\ &= -\frac{1}{4}\left(-\frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2F_{\mu\rho}F_{\nu}{}^{\rho}\right)\delta g^{\mu\nu} \ ,\end{aligned} \tag{1.4}$$

where in the last line we relabeled dummy indices so that all variations appear as $\delta g^{\mu\nu}$ and used the antisymmetry of the field strength to write $F^{\mu}{}_{\sigma} = -F_{\sigma}{}^{\mu}$. The stress-energy tensor is then given by the functional derivative of $\sqrt{-g}\,\mathcal{L}_M$ with respect to $\delta g^{\mu\nu}$:

$$\begin{aligned}T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_M)}{\delta g^{\mu\nu}}\\ &= -\frac{2}{\sqrt{-g}} \times -\frac{1}{4}\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 2F_{\mu\rho}F_{\nu}{}^{\rho}\right)\\ &= F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \ .\end{aligned} \tag{1.5}$$

This is the final result for the stress-energy tensor of Einstein-Maxwell theory.

2.) The Euler-Lagrange equation for the field A_μ can in full generality be written as

$$\frac{\partial\mathcal{L}_M}{\partial A_\mu} - \nabla_\nu\left(\frac{\partial\mathcal{L}_M}{\partial(\nabla_\nu A_\mu)}\right) = 0 \ . \tag{1.6}$$

The first term in the Euler-Lagrange equation is zero, since only derivatives of A_μ appear in the Lagrangian. Then, to compute the second term, we first note that

$$\frac{\partial F_{\rho\sigma}}{\partial(\nabla_\nu A_\mu)} = \frac{\partial(\nabla_\rho A_\sigma - \nabla_\sigma A_\rho)}{\partial(\nabla_\nu A_\mu)} = \delta^\nu_\rho\delta^\mu_\sigma - \delta^\nu_\sigma\delta^\mu_\rho \ . \tag{1.7}$$

Then, the product rule tells us that

$$\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\nabla_\nu A_\mu)} = 2F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial(\nabla_\nu A_\mu)} = 2F^{\rho\sigma}\left(\delta^\nu_\rho\delta^\mu_\sigma - \delta^\nu_\sigma\delta^\mu_\rho\right) = 2F^{\nu\mu} - 2F^{\mu\nu} = -4F^{\mu\nu} \ , \tag{1.8}$$

where the last equality comes from the antisymmetry of the field strength. Therefore the Euler-Lagrange equation becomes

$$0 = -\nabla_\nu\left(\frac{\partial\mathcal{L}_M}{\partial(\nabla_\nu A_\mu)}\right) = \frac{1}{4}\nabla_\nu\left(\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\nabla_\nu A_\mu)}\right) = -\nabla_\nu F^{\mu\nu} = \nabla_\nu F^{\nu\mu} \ . \tag{1.9}$$

It doesn't matter how we label the dummy and free indices in this expression, and so our final result for the equation of motion for the vector field A_μ in Einstein-Maxwell theory is

$$\nabla_\mu F^{\mu\nu} = 0 \ . \tag{1.10}$$

3.) First, we will explicitly write out the fully antisymmetrized quantity as

$$\nabla_{[\mu}F_{\nu\rho]} = \frac{1}{3!}\left(\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} - \nabla_\mu F_{\rho\nu} - \nabla_\rho F_{\nu\mu} - \nabla_\nu F_{\mu\rho}\right) \ . \tag{1.11}$$

The factor of 3! comes from the definition of the antisymmetrization. Now, since $F_{\mu\nu}$ is antisymmetric, we can actually see that many of these terms are equal, and this becomes

$$\nabla_{[\mu}F_{\nu\rho]} = \frac{1}{3}\left(\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu}\right) \equiv \frac{1}{3}G_{\mu\nu\rho} \ , \tag{1.12}$$

where to avoid having to keep track of the factor of three we defined $G_{\mu\nu\rho} = \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu}$. If we explicitly write out the covariant derivatives in terms of Christoffel symbols, this becomes

$$\begin{aligned} G_{\mu\nu\rho} = {} & \partial_\mu F_{\nu\rho} - \Gamma^\sigma_{\mu\nu}F_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}F_{\nu\sigma} \\ & + \partial_\nu F_{\rho\mu} - \Gamma^\sigma_{\nu\rho}F_{\sigma\mu} - \Gamma^\sigma_{\nu\mu}F_{\rho\sigma} \\ & + \partial_\rho F_{\mu\nu} - \Gamma^\sigma_{\rho\mu}F_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}F_{\mu\sigma} \ . \end{aligned} \tag{1.13}$$

By using the fact that the Christoffel symbols are symmetric in their lower indices and the field strengths are antisymmetric, you should be able to see that all terms with Christoffel symbols cancel, leaving us with

$$G_{\mu\nu\rho} = \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} \ . \tag{1.14}$$

Now, let's write out the field strengths in terms of Christoffel symbols and partial derivatives explicitly:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = (\partial_\mu A_\nu - \Gamma^\rho_{\mu\nu}A_\rho) - (\partial_\nu A_\mu - \Gamma^\rho_{\nu\mu}A_\rho) = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{1.15}$$

where again symmetry of the Christoffel symbols makes all explicit Christoffel terms cancel. Plugging this into (1.14), we find that

$$\begin{aligned} G_{\mu\nu\rho} &= \partial_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) + \partial_\nu\left(\partial_\rho A_\mu - \partial_\mu A_\rho\right) + \partial_\rho\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &= (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)A_\rho + (\partial_\nu\partial_\rho - \partial_\rho\partial_\nu)A_\mu + (\partial_\rho\partial_\mu - \partial_\mu\partial_\rho)A_\nu \\ &= 0 \ , \end{aligned} \tag{1.16}$$

where we regrouped terms in the second line and then in the third line we used the fact that ordinary partial derivatives commute. Thus we find that

$$0 = G_{\mu\nu\rho} \equiv \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} \ , \tag{1.17}$$

which in turn by (1.12) means that we have proven the Bianchi identity:

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \ . \tag{1.18}$$

A corollary to this is that we can always cyclically permute indices when derivatives act on the field strength:

$$\nabla_\mu F_{\nu\rho} = -\nabla_\nu F_{\rho\mu} - \nabla_\rho F_{\mu\nu} \ . \tag{1.19}$$

4.) Note: we will make use of the Bianchi identity below in this problem. In particular, as we will show in the next problem, the Bianchi identity tells us that

$$\nabla_\mu F_{\nu\rho} = -\nabla_\nu F_{\rho\mu} - \nabla_\rho F_{\mu\nu} \ . \tag{1.20}$$

With this in mind, the thing we want to compute is

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu \left(F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \\
&= (\nabla_\mu F^\mu{}_\rho) F^{\nu\rho} + F^\mu{}_\rho \nabla_\mu F^{\nu\rho} - \frac{1}{4} \nabla^\nu (F_{\rho\sigma} F^{\rho\sigma}) \\
&= (\nabla_\mu F^\mu{}_\rho) F^{\nu\rho} + F^\mu{}_\rho \nabla_\mu F^{\nu\rho} - \frac{1}{2} F_{\rho\sigma} \nabla^\nu F^{\rho\sigma} \ ,
\end{aligned} \tag{1.21}$$

where in the last line we used the product rule on the last term. Now, we note that the first term vanishes via the Maxwell equation of motion from problem 2, and so we are left with

$$\nabla_\mu T^{\mu\nu} = F^\mu{}_\rho \nabla_\mu F^{\nu\rho} - \frac{1}{2} F_{\rho\sigma} \nabla^\nu F^{\rho\sigma} = F_{\mu\rho} \left(\nabla^\mu F^{\nu\rho} - \frac{1}{2} \nabla^\nu F^{\mu\rho} \right) \ , \tag{1.22}$$

where the last equality is simply a relabelling of dummy indices, as well as using the fact that we can raise/lower pairs of contracted indices. We will now use the Bianchi identity on the second term to rewrite this as

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= F_{\mu\rho} \left(\nabla^\mu F^{\nu\rho} + \frac{1}{2} \nabla^\mu F^{\rho\nu} + \frac{1}{2} \nabla^\rho F^{\nu\mu} \right) \\
&= F_{\mu\rho} \left(\nabla^\mu F^{\nu\rho} - \frac{1}{2} \nabla^\mu F^{\nu\rho} + \frac{1}{2} \nabla^\rho F^{\nu\mu} \right) \\
&= \frac{1}{2} F_{\mu\rho} \left(\nabla^\mu F^{\nu\rho} + \nabla^\rho F^{\nu\mu} \right) \ ,
\end{aligned} \tag{1.23}$$

with the second equality coming from antisymmetry of the field strength. At this point, all we have to do is notice that $F_{\mu\rho}$ is antisymmetric under $\mu \leftrightarrow \rho$, while $\nabla^\mu F^{\nu\rho} + \nabla^\rho F^{\nu\mu}$ is symmetric under $\mu \leftrightarrow \rho$. Any antisymmetric tensor contracted with a symmetric tensor must vanish, and so we are simply left with

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{1.24}$$

2 de Sitter from Minkowski

5.) Proving that these coordinates satisfy the hyperboloid condition is relatively easy; you just need to square the appropriate terms, and then use the identities $\cos^2 x + \sin^2 x = \cosh^2 x - \sinh^2 x = 1$. To compute the induced metric, we use the chain rule to express

dt, dx, dy, and dz in terms of $d\rho$, du, and $d\psi$:

$$
\begin{aligned}
dt &= \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial \rho}d\rho = \frac{\sqrt{\alpha^2-\rho^2}}{\alpha}\cosh\left(\frac{u}{\alpha}\right)du - \frac{\rho}{\sqrt{\alpha^2-\rho^2}}\sinh\left(\frac{u}{\alpha}\right)d\rho\ , \\
dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial \rho}d\rho = \frac{\sqrt{\alpha^2-\rho^2}}{\alpha}\sinh\left(\frac{u}{\alpha}\right)du - \frac{\rho}{\sqrt{\alpha^2-\rho^2}}\cosh\left(\frac{u}{\alpha}\right)d\rho\ , \\
dy &= \frac{\partial y}{\partial \rho}d\rho + \frac{\partial y}{\partial \psi}d\psi = \sin\psi d\rho + \rho\cos\psi d\psi\ , \\
dz &= \frac{\partial z}{\partial \rho}d\rho + \frac{\partial z}{\partial \psi}d\psi = \cos\psi d\rho - \rho\sin\psi d\psi\ .
\end{aligned}
\tag{2.1}
$$

Now, we need to plug these into the metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. To do so, we first note that even though dt^2 and dx^2 will both have cross-terms of the form $dud\rho$, the cross-terms are the same for both, and thus they cancel when we take the difference. Therefore,

$$
\begin{aligned}
-dt^2 + dx^2 &= \frac{\alpha^2-\rho^2}{\alpha^2}\left[-\cosh^2\left(\frac{u}{\alpha}\right) + \sinh^2\left(\frac{u}{\alpha}\right)\right]du^2 \\
&\quad + \frac{\rho^2}{\alpha^2-\rho^2}\left[-\sinh^2\left(\frac{u}{\alpha}\right) + \cosh^2\left(\frac{u}{\alpha}\right)\right]d\rho^2 \\
&= -\frac{\alpha^2-\rho^2}{\alpha^2}du^2 + \frac{\rho^2}{\alpha^2-\rho^2}d\rho^2\ .
\end{aligned}
\tag{2.2}
$$

Similarly, both dy^2 and dz^2 have cross-terms of the form $d\rho d\psi$, but they cancel when we add the two quantities, leaving us with

$$
dy^2 + dz^2 = (\sin^2\psi + \cos^2\psi)d\rho^2 + \rho^2(\cos^2\psi + \sin^2\psi)d\psi^2 = d\rho^2 + \rho^2 d\psi^2\ . \tag{2.3}
$$

Adding everything up, the induced metric is

$$
\begin{aligned}
ds^2 &= -\frac{\alpha^2-\rho^2}{\alpha^2}du^2 + \frac{\rho^2}{\alpha^2-\rho^2}d\rho^2 + d\rho^2 + \rho^2 d\psi^2 \\
&= -\frac{\alpha^2-\rho^2}{\alpha^2}du^2 + \frac{\alpha^2}{\alpha^2-\rho^2}d\rho^2 + \rho^2 d\psi^2 \\
&= -\left(1-\frac{\rho^2}{\alpha^2}\right)du^2 + \left(1-\frac{\rho^2}{\alpha^2}\right)^{-1}d\rho^2 + \rho^2 d\psi^2\ .
\end{aligned}
\tag{2.4}
$$

This is the metric on the static patch of dS_3.

6.) First, recall that the Minkowski metric in spherical coordinates is

$$
ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \tag{2.5}
$$

The coordinate r is related to x, y, z by $r^2 = x^2 + y^2 + z^2$, and so we note that the hyperboloid condition is simply

$$
-t^2 + r^2 = \alpha^2\ . \tag{2.6}
$$

Then, it becomes very obvious that the closed slicing coordinates automatically satisfy this. It is also very straightforward to express dt and dr in terms of $d\tau$:

$$
dt = \frac{\partial t}{\partial \tau}d\tau = \cosh\left(\frac{\tau}{\alpha}\right)d\tau\ , \quad dr = \frac{\partial r}{\partial \tau}d\tau = \sinh\left(\frac{\tau}{\alpha}\right)d\tau\ . \tag{2.7}
$$

The induced metric is thus

$$
\begin{aligned}
ds^2 &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= -\cosh^2\left(\frac{\tau}{\alpha}\right)d\tau^2 + \sinh^2\left(\frac{\tau}{\alpha}\right)d\tau^2 + \alpha^2\cosh^2\left(\frac{\tau}{\alpha}\right)(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= -d\tau^2 + \alpha^2\cosh^2\left(\frac{\tau}{\alpha}\right)(d\theta^2 + \sin^2\theta\, d\phi^2)\ .
\end{aligned}
\tag{2.8}
$$

This is the metric for the closed slicing of dS_3.

7.) Consider the metric on the static patch

$$ds^2 = -\left(1 - \frac{\rho^2}{\alpha^2}\right) du^2 + \left(1 - \frac{\rho^2}{\alpha^2}\right)^{-1} d\rho^2 + \rho^2 d\psi^2 \ . \tag{2.9}$$

Consider a time-like slice at some fixed value of u of this metric. On this time-like slice, when $\rho = \alpha$, $g_{\rho\rho}$ blows up, and so we appear to have a singularity. We need to determine whether this is a true singularity, i.e. a singularity intrinsic to the full dS_3 manifold itself, or just some coordinate singularity that appears when we use the static patch metric.

To see that this is just a coordinate singularity, let's first revisit how the static patch coordinates are related to the original Minkowski coordinates:

$$\begin{aligned} t &= \sqrt{\alpha^2 - \rho^2} \sinh\left(\frac{u}{\alpha}\right) \ , \\ x &= \sqrt{\alpha^2 - \rho^2} \cosh\left(\frac{u}{\alpha}\right) \ , \\ y &= \rho \sin\psi \ , \\ z &= \rho \cos\psi \ . \end{aligned} \tag{2.10}$$

When we fix u to a constant and set $\rho = \alpha$, this corresponds to $t = x = 0$, while y and z are free to vary, subject to the condition that $y^2 + z^2 = \alpha^2$. This is the equation of a circle, and so the singular-looking $\rho = \alpha$ surface simply just corresponds to a circle in the full Minkowski spacetime, which we know should be a smooth and non-singular manifold.

Another way to see that there is no singularity is to consider the relationship between the coordinates (t, x, y, z) and the spherical coordinates (t, r, θ, ϕ):

$$x = r \sin\theta \cos\phi \ , \quad y = r \sin\theta \sin\phi \ , \quad z = r \cos\theta \ . \tag{2.11}$$

The only way to set $x = 0$ without forcing y and z to be zero is to set $\phi = \frac{\pi}{2}$ while fixing $r = \rho$. Then, if we remember that the closed slicing coordinates are defined by

$$\begin{aligned} t &= \alpha \sinh\left(\frac{\tau}{\alpha}\right) \ , \\ r &= \alpha \cosh\left(\frac{\tau}{\alpha}\right) \ , \end{aligned} \tag{2.12}$$

then we can see that setting $t = 0$ and $r = \rho$ is accomplished by setting $\tau = 0$. Therefore, the surface on the static patch defined by

$$u = \text{const.} \ , \quad \rho = \alpha \ , \quad \psi \in [0, 2\pi) \ , \tag{2.13}$$

is equivalent to the surface on the closed slicing defined by

$$\tau = 0 \ , \quad \theta \in [0, \pi) \ , \quad \phi = \frac{\pi}{2} \ . \tag{2.14}$$

Importantly, the closed slicing metric is smooth and well-behaved on this surface! In fact, the metric on this surface simply becomes $ds^2 = \alpha^2 d\theta^2$, which is again the metric on a circle.

Therefore, because we have a set of coordinates on dS_3 where this surface is actually well-behaved and smooth, there cannot be a true singularity at $\rho = \alpha$. This is exactly analagous to how there is a coordinate singularity at $r = 2GM$ in the Schwarzschild metric, but by going into Kruskal coordinates we can see that this location is actually a smooth and non-singular region that we refer to as the black hole horizon. In fact, the surface defined by $\rho = \alpha$ is referred to as the *cosmological horizon* of dS_3.

8.) The $(d+1)$-dimensional Minkowski metric in spherical coordinates is simply

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2 \ , \tag{2.15}$$

where $d\Omega_{d-1}^2$ is the metric on S^{d-1}. The d-dimensional hyperboloid that defines dS_d is specified in spherical coordinates by

$$-dt^2 + dr^2 = \alpha^2 \ , \tag{2.16}$$

for some positive constant α, just like we found earlier in the $d = 3$ case. We can easily solve this hyperboloid condition with exactly the same closed slicing parameterization as before:

$$\begin{aligned} t &= \alpha \sinh\left(\frac{\tau}{\alpha}\right) \ , \\ r &= \alpha \cosh\left(\frac{\tau}{\alpha}\right) \ , \end{aligned} \tag{2.17}$$

with the coordinates on S^{d-1} unconstrained. The induced metric in this closed slicing is then simply

$$ds^2 = -d\tau^2 + \alpha^2 \cos^2\left(\frac{\tau}{\alpha}\right) d\Omega_{d-1}^2 \ . \tag{2.18}$$

This is the closed slicing metric on dS_d, for any integer value of d.

We can also construct the static patch on dS_d with just a little bit of extra work. First, in normal Cartesian coordinates, we can write the $(d+1)$-dimensional Minkowski metric as

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + \ldots + dx_d^2 \ . \tag{2.19}$$

The hyperboloid submanifold of size α is then given by

$$-t^2 + \sum_{i=1}^{d} dx_i^2 = \alpha^2 \ . \tag{2.20}$$

To parameterize this hyperboloid, we will introduce a set of coordinates (u, ρ, z_i), with $2 \leq i \leq d$, such that

$$\begin{aligned} t &= \sqrt{\alpha^2 - \rho^2} \sinh\left(\frac{u}{\alpha}\right) \ , \\ x_1 &= \sqrt{\alpha^2 - \rho^2} \cosh\left(\frac{u}{\alpha}\right) \ , \\ x_i &= \rho z_i \quad (2 \leq i \leq d) \ , \end{aligned} \tag{2.21}$$

subject to the condition that

$$\sum_{i=2}^{d} z_i = 1 \ . \tag{2.22}$$

Through a straightforward computation, you should be able to verify that this satisfies the hyperboloid condition. Moreover, since the z_i sum in quadrature to one, they should be thought of as Cartesian coordinates on S^{d-2}, i.e. a $(d-2)$-sphere with unit radius, and thus we can choose them such that

$$\sum_{i=2}^{d} dz_i^2 = d\Omega_{d-2}^2 \ . \tag{2.23}$$

If we put this all together, you can show (by a trivial extension of our computation in problem 5) that the induced metric for this d-dimensional static patch is

$$ds^2 = -\left(1 - \frac{\rho^2}{\alpha^2}\right) du^2 + \left(1 - \frac{\rho^2}{\alpha^2}\right)^{-1} d\rho^2 + \rho^2 d\Omega_{d-2}^2 \ . \tag{2.24}$$

Just as in the $d = 3$ case, there appears to be a singularity at $\rho = \alpha$ in the static patch on dS_d. And, just like in the $d = 3$ case, the resolution of this issue is to consider this surface in the closed slicing, wherein the surface becomes smooth and singularity-free. Thus, there is yet again a cosmological horizon on this static patch.